

Weight Function Approach to Study a Crack Propagating Along a Bimaterial Interface Under Arbitrary Loading in Orthotropic Solids

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1 Introduction

This paper considers a steady-state crack propagating along an interface between dissimilar anisotropic materials under an asymmetric load. As a model, a semi-infinite crack propagating at a constant speed along a perfect interface between two anisotropic half planes is analysed. Such a problem cannot be directly resolved using the approach developed by Ting (1980s) and Sih (1960s) by using Stroh formalism as was seen in Stroh (1950s). This approach should be adjusted to tackle non-symmetric loading. This has been done for the static problem in Morini (2012) where the Stroh formalism was combined with the weight function technique. In particular, they found analytic expressions for the stress intensity factors. The main aim of this paper is to extend these results for a steady-state crack propagation and to propose a method for computing important fracture mechanics parameters (e.g. stress intensity factors, energy release rates) efficiently for an arbitrary loading that is non-smooth and does not necessarily have to be decreasing rapidly at infinity. We consider, as an example, only orthotropic materials while the approach can be extended to monoclinic materials as well. In addition, the second order asymptotic terms are also found for an arbitrary load, which is essential when performing perturbation analysis for the propagating interfacial crack. Finally, we present results of numerical computations for a specific asymmetric load and discuss them in the context of fracture mechanics.

2 Preliminary Results

In this section the mathematical model considered will be introduced along with the main mathematical results used for further analysis of the problem. A semi-infinite crack propagating at a constant speed, v , along a perfect interface between two semi-infinite anisotropic materials is considered. The crack is said to be occupying the region $x_1 - vt < 0, x_2 = 0$ as illustrated in Figure 1.

The traction on the crack faces is known and is said to be given by the known functions $\sigma_{2i}(x_1 - vt, 0^\pm) = p_j^\pm(x_1 - vt)$ for $x_1 - vt < 0$ and body forces are assumed to be zero. No restrictions are imposed on the loading considered in this paper, that is: the loading need not vanish at infinity or the crack tip.

Section 2.1 introduces the previous results obtained for a static crack. These results include expressions for the traction along the interface and the displacement jump across the crack along

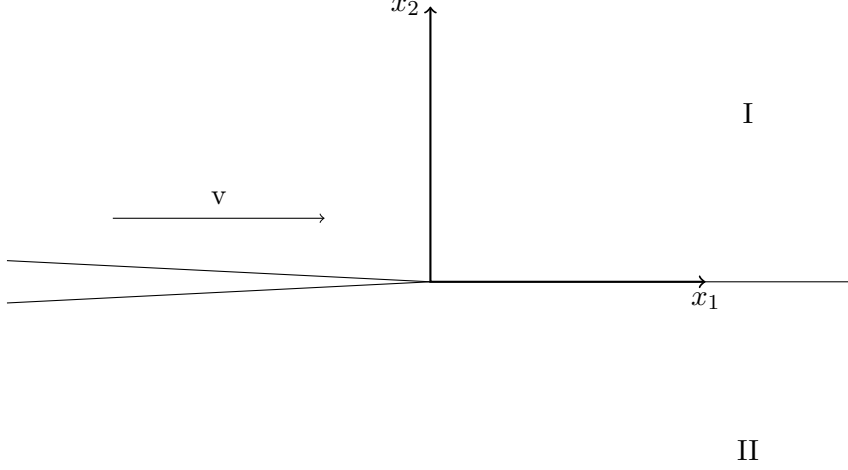


Figure 1: Geometry

with their asymptotic expansions in the neighbourhood of the crack tip. Stress intensity factors are also presented along with weight functions and the Betti formula for a static crack. Section 2.2 discusses the dynamic crack and makes note of similar results as those presented for the static crack. Results that have been found for only the static problem that are going to be derived for the steady state in this paper are also noted.

2.1 Static Crack

2.1.1 Stroh Analysis

The Stroh formalism, as used in Stroh (1962), is used in order to find expressions for both the displacement field and the stresses in the material. It makes use of Hooke's law, given by

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} = C_{ijkl}\frac{\partial u_k}{\partial x_l}, \quad \text{for } i, j, k, l = 1, 2, \quad (1)$$

where σ is the stress, ϵ is the strain and C is the stiffness tensor for the material. Furthermore, the following relationship relating the stress and displacement is also used

$$\sum_{j,k,l=1}^2 \frac{\partial \sigma_{ij}}{\partial x_j} = 0. \quad (2)$$

Combining (1) and (2) it is seen that

$$C_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} = 0. \quad (3)$$

A solution for the displacement field is assumed in the form $u_i = A_i f(x_1 + px_2)$ and this in conjunction with (3) gives the following eigenvalue problem

$$(C_{i1k1} + pC_{i1k2} + pC_{i2k1} + p^2C_{i2k2})A_k = 0. \quad (4)$$

In matrix form this is written as

$$(\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T})\mathbf{A} = 0, \quad (5)$$

where \mathbf{Q} , \mathbf{R} and \mathbf{T} all depend on elastic constants for the given material. Solving this eigenvalue problem to find the eigenvalues p and eigenvectors A_i the following relationships for the displacement vector and traction vector are obtained (Suo, 1990)

$$\mathbf{u}(x_1, x_2) = \mathbf{A}\mathbf{f}(z) + \overline{\mathbf{A}\mathbf{f}(z)}, \quad (6)$$

$$\mathbf{t}(x_1, x_2) = \mathbf{L}\mathbf{f}'(z) + \overline{\mathbf{L}\mathbf{f}'(z)}, \quad (7)$$

where $z = x_1 + px_2$. Here the eigenvalues, p , are taken to be those with positive imaginary part and the matrix \mathbf{A} is constructed from the eigenvectors of (5). The vector $\mathbf{f}(z)$ is given by $f_i = f(x_1 + p_ix_2)$. The matrix \mathbf{L} is given by

$$\mathbf{L} = (\mathbf{R}^T + p\mathbf{T})\mathbf{A}.$$

At this stage the matrix $\mathbf{B} = i\mathbf{A}\mathbf{L}^{-1}$ is introduced, where $i^2 = -1$. The bimaterial matrices \mathbf{H} and \mathbf{W} are also introduced, where

$$\mathbf{H} = \mathbf{B}_I + \bar{\mathbf{B}}_{II}, \quad \mathbf{W} = \mathbf{B}_I - \bar{\mathbf{B}}_{II},$$

where the subscript I or II determines which material the matrix relates to.

The analysis performed in Suo (1990) considered the same physical problem as shown in figure 1, with $v = 0$, with zero traction conditions imposed on the crack faces and continuous traction and displacement across the interface ($x_1 > 0$). The following results are found to be true along the real axis (Suo, 1990)

$$\begin{aligned} \mathbf{h}^+(x_1) + \bar{\mathbf{H}}^{-1}\mathbf{H}\mathbf{h}^-(x_1) &= \mathbf{t}(x_1), \quad 0 < x_1 < \infty, \\ \mathbf{h}^+(x_1) + \bar{\mathbf{H}}^{-1}\mathbf{H}\mathbf{h}^-(x_1) &= 0, \quad -\infty < x_1 < 0. \end{aligned} \quad (8)$$

Here, $\mathbf{h}(z)$ is a function assumed in the form $\mathbf{h}(z) = \mathbf{w}z^{-\frac{1}{2}+i\epsilon}$. The branch cut of $\mathbf{h}(z)$ is placed along the negative real axis and both \mathbf{w} and ϵ are derived from the eigenvalue problem (Suo, 1990)

$$\bar{\mathbf{H}}\mathbf{w} = e^{2\pi\epsilon}\mathbf{H}\mathbf{w}. \quad (9)$$

An expression for the traction ahead of the crack was found in Suo (1990)

$$\mathbf{t}(x_1) = \frac{1}{\sqrt{2\pi x_1}} \text{Re}(Kx_1^{i\epsilon}\mathbf{w}), \quad (10)$$

where \mathbf{w} is found using the eigenvalue problem (9). $K = K_I + iK_{II}$ is known as the stress intensity factor and includes the contributing factors to the traction of both mode I and mode II .

The displacement jump across the crack, defined as $[\mathbf{u}]$, was also found in Suo (1990), where $[\cdot]$ is given by

$$[f](x) = f(x, 0^+) - f(x, 0^-). \quad (11)$$

For $x_1 < 0$ it was found that

$$[\mathbf{u}](x_1) = \left(\frac{2(-x_1)}{\pi}\right)^{\frac{1}{2}} \frac{(\mathbf{H} + \bar{\mathbf{H}})}{\cosh \pi\epsilon} \text{Re} \left(\frac{K(-x_1)^{i\epsilon}\mathbf{w}}{1 + 2i\epsilon} \right). \quad (12)$$

The asymptotic expansions of the physical traction and the jump in displacement across the interface as $x \rightarrow 0$ can be written as follows (Morini, 2012)

$$[\mathbf{u}](x_1) = \frac{(-x_1)^{\frac{1}{2}}}{\sqrt{2\pi}} \mathcal{U}(x_1) \mathbf{K} + \frac{(-x_1)^{\frac{3}{2}}}{\sqrt{2\pi}} \mathcal{U}(x_1) \mathbf{Y}_2 + \frac{(-x_1)^{\frac{5}{2}}}{\sqrt{2\pi}} \mathcal{U}(x_1) \mathbf{Y}_3 + \mathcal{O}((-x_1)^{\frac{7}{2}}), \quad (13)$$

$$\mathbf{t}(x_1) = \frac{x_1^{-\frac{1}{2}}}{2\sqrt{2\pi}}\mathcal{T}(x_1)\mathbf{K} + \frac{x_1^{\frac{1}{2}}}{2\sqrt{2\pi}}\mathcal{T}(x_1)\mathbf{Y}_2 + \frac{x_1^{\frac{3}{2}}}{2\sqrt{2\pi}}\mathcal{T}(x_1)\mathbf{Y}_3 + \mathcal{O}(x_1^{\frac{5}{2}}), \quad (14)$$

where $\mathbf{K} = [K, \bar{K}]$ and $\mathbf{Y}_i = [Y_i, \bar{Y}_i]$. Y_i are constants derived in the same manner as the stress intensity factor K in order to find further terms in the asymptotic expansions. The matrices $\mathcal{U}(x_1)$ and $\mathcal{T}(x_1)$ are represented as follows

$$\mathcal{U}(x_1) = \frac{2(\mathbf{H} + \bar{\mathbf{H}})}{\cosh \pi \epsilon} \left[\frac{\mathbf{w}(-x_1)^{i\epsilon}}{1 + 2i\epsilon}, \frac{\bar{\mathbf{w}}(-x_1)^{-i\epsilon}}{1 - 2i\epsilon} \right], \quad \mathcal{T}(x_1) = 2 [\mathbf{w}x_1^{i\epsilon}, \bar{\mathbf{w}}x_1^{-i\epsilon}]. \quad (15)$$

2.1.2 Stress Intensity Factors

An explicit formula for computing the stress intensity factor for symmetric loading was given in Suo (1990). It was shown that

$$\mathbf{K} = - \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \cosh \pi \epsilon \int_{-\infty}^0 (-x_1)^{-\frac{1}{2}-i\epsilon} \mathbf{p}_1(x_1) dx_1, \quad (16)$$

where the vector $\mathbf{p}_1(x_1)$ is related to the applied traction $\mathbf{p}(x_1)$ in the following way

$$\mathbf{p}_1 = \frac{\bar{\mathbf{w}}^T \mathbf{H} \mathbf{p}}{\bar{\mathbf{w}}^T \mathbf{H} \mathbf{w}}.$$

2.1.3 Energy Release Rate

A key component in the analysis of fracture mechanics is the determination of the energy release rate (ERR) when a unit area of interface is cracked. An expression was found for the ERR, denoted G , in Irwin (1957)

$$G = \frac{1}{2\Delta} \int_0^\Delta \mathbf{t}^T(\Delta - r) [\mathbf{u}](r) dr, \quad (17)$$

where Δ is an arbitrary length scale. It was found in Suo (1990) that using (10) and (12) that the following expression was found for the energy release rate

$$G = \frac{\bar{\mathbf{w}}^T (\mathbf{H} + \bar{\mathbf{H}}) \mathbf{w} |K|^2}{4 \cosh^2(\pi \epsilon)} \quad (18)$$

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2.1.4 Weight Functions

The weight function \mathbf{U} is now defined in the same vein as Willis (1995). $\mathbf{U} = (U_1, U_2)^T$ is the singular displacement field that is obtained in the problem where the static crack occupies the part of the axis with $x_1 > 0$ is now considered. Therefore \mathbf{U} is discontinuous over the positive portion of the real axis. The symmetric part of the weight function is given by $[\mathbf{U}](x_1)$ and skew-symmetric part of the weight function is given by $\langle \mathbf{U} \rangle(x_1)$, where $\langle \cdot \rangle$ is defined in the usual manner

$$\langle f \rangle(x) = \frac{1}{2} [f(x, 0^+) + f(x, 0^-)]. \quad (19)$$

The traction field associated with the displacement field, \mathbf{U} , is denoted as $\mathbf{\Upsilon} = (\Upsilon_1, \Upsilon_2)^T$ and is said to be continuous over the interface ($x_1 < 0$) and the zero traction condition is imposed on the crack faces. Therefore, the following relationship stands for this problem, as seen in Morini (2012)

$$\mathbf{h}_+(x_1) + \bar{\mathbf{H}}^{-1} \mathbf{H} \mathbf{h}_-(x_1) = 0 \quad \text{for } x_1 > 0, \quad (20)$$

$$\mathbf{h}_+(x_1) + \bar{\mathbf{H}}^{-1} \mathbf{H} \mathbf{h}_-(x_1) = \Upsilon(x_1) \quad \text{for } x_1 < 0. \quad (21)$$

A solution for $\mathbf{h}(z)$ is assumed in the form

$$\mathbf{h}(z) = \mathbf{v} z^{-\frac{3}{2} + i\epsilon}, \quad (22)$$

where the branch cut is now said to be along the positive x_1 -axis. This gives the eigenvalue problem

$$\bar{\mathbf{H}} \mathbf{v} = e^{-2\pi\epsilon} \mathbf{H} \mathbf{v}. \quad (23)$$

\mathbf{H} is positive definite hermitian and therefore it is clear, by comparing (23) with (9), that $\mathbf{v} = \bar{\mathbf{w}}$. Therefore the singular traction in the steady state has the form (Morini, 2012)

$$\Upsilon(x) = \frac{(-x)^{-\frac{3}{2}}}{\sqrt{2\pi}} \text{Re}(R(-x_1)^{i\epsilon} \bar{\mathbf{w}}), \quad (24)$$

where $R = R_1 + iR_2$ is a complex number in a similar fashion to the stress intensity factor for the physical problem.

An expression relating the symmetric and skew-symmetric weight functions was found in Morini (2012) following from the work seen in Piccolroaz (2007)

$$[\hat{U}]^+(\chi) = \frac{1}{|\chi|} (i \text{sign}(\chi) \text{Im}(\mathbf{H}) - \text{Re}(\mathbf{H})) \hat{\Upsilon}^-(\chi), \quad (25)$$

$$\langle \hat{U} \rangle(\chi) = \frac{1}{2|\chi|} (i \text{sign}(\chi) \text{Im}(\mathbf{W}) - \text{Re}(\mathbf{W})) \hat{\Upsilon}^-(\chi), \quad (26)$$

where the supercripts \pm denotes whether the function is analytic in the upper or lower half plane respectively. The $\hat{\cdot}$ denotes the Fourier transform, defined as

$$\hat{f}(\chi) = \int_{-\infty}^{\infty} f(x) e^{i\chi x} dx.$$

The Wiener-Hopf problem shown in (25) and (26) was solved in Morini (2012) and the corresponding parts of the weight function were shown there and will not be repeated in this paper for a static crack. However, later in this paper, equations (25) and (26) will be presented for the steady state despite the fact that they are obtained using the exact method used in Morini (2012). This is done because the representation used in this paper has a different branch cut to that in Morini (2012) which enables the calculation of further terms in the asymptotic expansions of the displacement and stress fields. Moreover, it is possible to obtain the equivalent result for a static crack by adopting the same method introduced for the steady state in this paper and setting $v = 0$.

2.1.5 Betti Formula

It was mentioned previously that there are now two displacement fields to consider; the physical displacement \mathbf{u} and the singular displacement \mathbf{U} . However, \mathbf{U} is discontinuous for $x_1 > 0$ whereas \mathbf{u} is discontinuous for $x_1 < 0$. Also considered is the traction associated with \mathbf{U} given by $\mathbf{\Upsilon}$ which is continuous when $x_1 < 0$ and the traction \mathbf{t} associated with \mathbf{u} which is continuous when $x_1 > 0$. It is possible to write an expression for the physical traction as (Morini, 2012)

$$\sigma(x_1, x_2)|_{x_2=0^+} = \mathbf{p}_+(x_1) + \mathbf{t}(x_1), \quad \sigma(x_1, x_2)|_{x_2=0^-} = \mathbf{p}_-(x_1) + \mathbf{t}(x_1). \quad (27)$$

Following the approach used in Willis (1995) and Piccolroaz (2007) the Betti formula is used to find a relationship between the weight functions and the physical solutions. It was seen that

$$[\hat{U}]^{+T} \mathcal{R} \hat{\mathbf{t}}^+ - \hat{\mathbf{\Upsilon}}^{-T} \mathcal{R} [\hat{\mathbf{u}}]^- = -[\hat{U}]^{+T} \mathcal{R} \langle \hat{\mathbf{p}} \rangle - \langle \hat{\mathbf{U}} \rangle^T \mathcal{R} [\hat{\mathbf{p}}], \quad (28)$$

where

$$\mathcal{R} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

. Here, $\langle \mathbf{p} \rangle$ and $[\mathbf{p}]$ refer to the symmetric and skew-symmetric parts of the loading respectively.

Further work performed in Piccolroaz (2007) and Morini (2012), combining (25), (26) and (28), found an explicit expression for finding the stress intensity factor, \mathbf{K} , using the weight functions and the loading applied on the crack faces. The following expression was obtained

$$\mathbf{K} = \frac{1}{2\pi i} \mathcal{Z}_1^{-1} \int_{-\infty}^{\infty} [\hat{\mathbf{U}}]^{+T}(\tau) \mathbf{R} \langle \hat{\mathbf{p}} \rangle(\tau) + \langle \hat{\mathbf{U}} \rangle^T(\tau) \mathbf{R} [\hat{\mathbf{p}}](\tau) d\tau, \quad (29)$$

where \mathcal{Z}_1 is a constant matrix derived from the asymptotic representation of (28). The matrix \mathcal{Z}_1 used in Morini (2012) uses different notation to that used in this paper but the same matrix can be found using results in section 3 with $v = 0$.

In Suo (1990) symmetric loading was considered and therefore $[\mathbf{p}] = 0$. It can easily be shown that both expressions for \mathbf{K} , (16) and (29), are equivalent. [See discussion in section 3.](#)

Following the method developed in Piccolroaz (2007) and Morini (2012) an expression for further asymptotic coefficients can be found depending on whether the applied loading is smooth and has a Fourier transform that vanishes at a fast enough rate at infinity. If this is the case the general expression for the asymptotic coefficients can be found using the equation

$$\mathbf{Y}_j = \frac{1}{2\pi i} \mathcal{Z}_j^{-1} \int_{-\infty}^{\infty} \tau^{j-1} \{ [\hat{\mathbf{U}}]^{+T}(\tau) \mathbf{R} \langle \hat{\mathbf{p}} \rangle(\tau) + \langle \hat{\mathbf{U}} \rangle^T(\tau) \mathbf{R} [\hat{\mathbf{p}}](\tau) \} d\tau. \quad (30)$$

Here, \mathcal{Z}_j is also derived from the asymptotic representation of (28) and is found for the steady state in section 3 of this paper and can be found for the static crack by setting $v = 0$ in the steady state formulation.

2.2 Steady-State Case

The steady state problem of figure 1 was considered in Yang (1991) and Yu (2000). In this paper it is assumed that the crack velocity v is constant and is lower than the Rayleigh wave speeds of both materials I and II .

2.2.1 Stroh Formalism and Suo Results for the Steady State

When the crack is moving at the velocity v equation (2) becomes

$$\sum_{j,k,l=1}^2 \frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad (31)$$

where ρ is the material density. Using the same method as used for the static crack (1) and (31) are combined to give

$$C_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} = \rho \frac{\partial^2 u_i}{\partial t^2}. \quad (32)$$

A new coordinate system is now introduced: $(\tilde{x}_1 = x_1 - vt, \tilde{x}_2 = x_2)$. The following relationship is therefore found in this new coordinate system

$$\tilde{C}_{ijkl} \frac{\partial^2 u_k}{\partial \tilde{x}_j \partial \tilde{x}_l} = 0, \quad (33)$$

where $\tilde{C}_{ijkl} = C_{ijkl} - \rho v^2 \delta_{ik} \delta_{1j} \delta_{1l}$.

From this stage, for convenience, the steady state coordinates will be written as $\tilde{x}_1 = x$ and $\tilde{x}_2 = y$. A solution is now assumed in the form $u_i = A_i f(x + py)$ the eigenvalue problem (5) is once again obtained. However, the matrix \mathbf{Q} now relies on both the material constants and the crack velocity. The matrices \mathbf{R} and \mathbf{T} are identical to those in (5).

Using the same methods as used for the static crack all equations from (5) to (16) still hold for the steady state with the coordinates (x, y) . It is noted that the variables $\mathbf{A}, \mathbf{L}, \mathbf{B}, \mathbf{H}, \mathbf{w}$ and ϵ depend on the crack velocity, as well as the elastic coefficients of the materials, for the steady state.

2.2.2 Weight Function for the Steady-State Case

The derivation of the weight function, \mathbf{U} , for the steady-state coordinate system can be found using the same methods used for the static coordinate system, shown in Morini (2012). A different representation is used in this paper allows a more convenient approach to the computation of higher order terms in the asymptotic expansions of the displacement and stress fields.

Similarly to the static case seen in Morini (2012) the weight function, \mathbf{U} , is obtained as the singular displacement field that is obtained in the problem where the crack occupies the part of the axis with $x > 0$ is considered in order to obtain weight functions. The symmetric part and skew-symmetric part of the weight function are once again defined as $[\mathbf{U}](x)$ and $\langle \mathbf{U} \rangle(x)$ respectively. The traction vector, denoted Υ , corresponding to this singular displacement field is assumed to be continuous over the interface ($x < 0$) and the traction free condition is imposed on the crack faces. Using the same methods as were used in Morini (2012) it can be shown that equations (25) and (26) still hold for the steady state.

Expressions for the symmetric and skew-symmetric parts of the weight function, \mathbf{U} , for orthotropic bimaterials are now found. To do this the associated traction, Υ , must be found. The method used for the static crack in Morini (2012) is used (with the results (36) and (37) written in a different way). Substituting the value for \mathbf{w} from (78) into (24), and saying $R = 1$, yields the following result for $-\infty < x < 0$

$$\Upsilon^1(x) = \frac{(-x)^{-\frac{3}{2}}}{2\sqrt{2\pi}} \left(\frac{i[(-x)^{i\epsilon} - (-x)^{-i\epsilon}]}{\sqrt{\frac{H_{11}}{H_{22}}}[(-x)^{i\epsilon} + (-x)^{-i\epsilon}]} \right). \quad (34)$$

Continuing to follow the method from Piccolroaz (2009) the same vectors is computed with $R = i$

$$\Upsilon^2(x) = \frac{(-x)^{-\frac{3}{2}}}{2\sqrt{2\pi}} \left(\frac{-[(-x)^{i\epsilon} + (-x)^{-i\epsilon}]}{i\sqrt{\frac{H_{11}}{H_{22}}}[(-x)^{i\epsilon} - (-x)^{-i\epsilon}]} \right). \quad (35)$$

The Fourier transforms of these two vectors is now computed. The branch cut for these vectors is situated along the positive real axis and polar coordinates with angle between -2π and 0 are taken. The identity

$$\int_0^\infty t^b e^{-at} dt = \frac{\Gamma(b+1)}{a^{b+1}} \quad \text{for } b > -1,$$

is also used. The Fourier transforms obtained are

$$\hat{\Upsilon}^{1-}(\chi) = \frac{(i\chi)^{\frac{1}{2}}\sqrt{2}}{(1+4\epsilon^2)\sqrt{\pi}} \left(\frac{i \left[(-\frac{1}{2} - i\epsilon)\Gamma(\frac{1}{2} + i\epsilon)(i\chi)^{-i\epsilon} - (-\frac{1}{2} + i\epsilon)\Gamma(\frac{1}{2} - i\epsilon)(i\chi)^{i\epsilon} \right]}{\sqrt{\frac{H_{11}}{H_{22}}} \left[(-\frac{1}{2} - i\epsilon)\Gamma(\frac{1}{2} + i\epsilon)(i\chi)^{-i\epsilon} + (-\frac{1}{2} + i\epsilon)\Gamma(\frac{1}{2} - i\epsilon)(i\chi)^{i\epsilon} \right]} \right), \quad (36)$$

$$\hat{\Upsilon}^{2-}(\chi) = \frac{(i\chi)^{\frac{1}{2}}\sqrt{2}}{(1+4\epsilon^2)\sqrt{\pi}} \left(\frac{- \left[(-\frac{1}{2} - i\epsilon)\Gamma(\frac{1}{2} + i\epsilon)(i\chi)^{-i\epsilon} + (-\frac{1}{2} + i\epsilon)\Gamma(\frac{1}{2} - i\epsilon)(i\chi)^{i\epsilon} \right]}{i\sqrt{\frac{H_{11}}{H_{22}}} \left[(-\frac{1}{2} - i\epsilon)\Gamma(\frac{1}{2} + i\epsilon)(i\chi)^{-i\epsilon} - (-\frac{1}{2} + i\epsilon)\Gamma(\frac{1}{2} - i\epsilon)(i\chi)^{i\epsilon} \right]} \right), \quad (37)$$

where $\Gamma(\cdot)$ is the gamma function and the branch cut of Υ^- is situated along the positive imaginary axis. Note that the expressions (36) and (37) are written using a different representation than was used in Morini (2012). The reason behind this will become clearer in section 3.

The Fourier transforms (25) and (26) can now be computed, for $\chi \in \mathbb{R}$, with the aid of equations (77) and (79).

$$\begin{aligned} [\hat{U}]^+(\chi) &= \frac{1}{|\chi|} \left[\begin{pmatrix} 0 & -i\beta \text{sign}(\chi) \sqrt{H_{11}H_{22}} \\ i\beta \text{sign}(\chi) \sqrt{H_{11}H_{22}} & 0 \end{pmatrix} - \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix} \right] \hat{\Upsilon}^-(\chi), \\ &= \frac{1}{|\chi|} \begin{pmatrix} -H_{11} & -i\beta \text{sign}(\chi) \sqrt{H_{11}H_{22}} \\ i\beta \text{sign}(\chi) \sqrt{H_{11}H_{22}} & -H_{22} \end{pmatrix} \hat{\Upsilon}^-(\chi), \end{aligned} \quad (38)$$

$$\begin{aligned} \langle \hat{U} \rangle(\chi) &= \frac{1}{2|\chi|} \left[\begin{pmatrix} 0 & i\gamma \text{sign}(\chi) \sqrt{H_{11}H_{22}} \\ -i\gamma \text{sign}(\chi) \sqrt{H_{11}H_{22}} & 0 \end{pmatrix} - \begin{pmatrix} \delta_1 H_{11} & 0 \\ 0 & \delta_2 H_{22} \end{pmatrix} \right] \hat{\Upsilon}^-(\chi), \\ &= \frac{1}{2|\chi|} \begin{pmatrix} -\delta_1 H_{11} & i\gamma \text{sign}(\chi) \sqrt{H_{11}H_{22}} \\ -i\gamma \text{sign}(\chi) \sqrt{H_{11}H_{22}} & -\delta_2 H_{22} \end{pmatrix} \hat{\Upsilon}^-(\chi), \end{aligned} \quad (39)$$

where branch cuts are now situated along the negative imaginary axis.

2.2.3 Betti Formula for the Steady State

It was shown in Willis (1995) that the Betti formula still holds for the steady state crack in isotropic materials. Using the same method it can be shown that the Betti formula still holds for the moving coordinate system in anisotropic materials. Therefore, the following expressions are found along the upper and lower parts of the real axis, respectively

$$\int_{-\infty}^{\infty} \{ \mathbf{U}^T(x' - x, 0^+) \mathcal{R}\sigma(x, 0^+) - \mathbf{\Upsilon}^T(x' - x, 0^+) \mathcal{R}\mathbf{u}(x, 0^+) \} dx = 0, \quad (40)$$

$$\int_{-\infty}^{\infty} \{ \mathbf{U}^T(x' - x, 0^-) \mathcal{R}\sigma(x, 0^-) - \mathbf{\Upsilon}^T(x' - x, 0^-) \mathcal{R}\mathbf{u}(x, 0^-) \} dx = 0. \quad (41)$$

The homogeneous case of (8) is now considered. Combined with the applied traction on the crack faces, $\mathbf{p}(x)$, the following expressions for traction are obtained

$$\sigma_{2i}(x, y = 0^+) = \mathbf{p}^+(x) + \mathbf{t}(x), \quad \sigma_{2i}(x, y = 0^-) = \mathbf{p}^-(x) + \mathbf{t}(x). \quad (42)$$

Subtracting (41) from (40) and using (27) along the definition of the symmetric and skew-symmetric parts of the weight function the following formula is obtained

$$\begin{aligned} &\int_{-\infty}^{\infty} \{ [\mathbf{U}]^T(x' - x) \mathcal{R}\mathbf{t}(x) - \mathbf{\Upsilon}^T(x' - x, 0) \mathcal{R}[\mathbf{u}](x) \} dx \\ &= - \int_{-\infty}^{\infty} \{ [\mathbf{U}]^T(x' - x) \mathcal{R}\langle \mathbf{p} \rangle(x) + \langle \mathbf{U} \rangle^T(x' - x) \mathcal{R}[\mathbf{p}](x) \} dx, \end{aligned} \quad (43)$$

Using the Fourier convolution theorem the following identity, which relates the Fourier transforms of the weight functions and the solutions of the physical problem, is obtained (Piccolroaz, 2007), (Morini, 2012)

$$[\hat{\mathbf{U}}]^{+T} \mathcal{R}\hat{\mathbf{t}}^+ - \hat{\mathbf{\Upsilon}}^{-T} \mathcal{R}[\hat{\mathbf{u}}]^- = -[\hat{\mathbf{U}}]^{+T} \mathcal{R}\langle \hat{\mathbf{p}} \rangle - \langle \hat{\mathbf{U}} \rangle^T \mathcal{R}[\hat{\mathbf{p}}], \quad (44)$$

where the \pm denotes whether the transform is analytic in the upper or lower half plane.

3 Evaluation of the Coefficients in the Asymptotic Expansion of the Displacement and Stress Fields

3.1 Determination of the Stress Intensity Factor

It is now possible to develop a method in order to find the stress intensity factor for orthotropic materials, similar to that seen for the static crack in Morini (2012). In the case of orthotropic materials the matrix $\mathcal{T}(x)$ in equation (14) is given by

$$\mathcal{T}(x) = \begin{pmatrix} -ix^{i\epsilon} & ix^{-i\epsilon} \\ \sqrt{\frac{H_{11}}{H_{22}}}x^{i\epsilon} & \sqrt{\frac{H_{11}}{H_{22}}}x^{-i\epsilon} \end{pmatrix}. \quad (45)$$

Note that this result is equivalent to (15) with the known value of \mathbf{w} inserted. The Fourier transform of this expansion is computed in order to find the asymptotic expansion as $\chi \rightarrow \infty$, with $\text{Im}(\chi) \in (0, \infty)$. The result is

$$\hat{t}(\chi) = \frac{(-i\chi)^{-\frac{1}{2}}}{2\sqrt{2\pi}} \mathfrak{T}_1(\chi) \mathbf{K} + \frac{(-i\chi)^{-\frac{3}{2}}}{2\sqrt{2\pi}} \mathfrak{T}_2(\chi) \mathbf{Y} + \mathcal{O}((\chi)^{-\frac{5}{2}}), \quad (46)$$

where

$$\mathfrak{T}_1(\chi) = \begin{pmatrix} -i(-i\chi)^{-i\epsilon}\Gamma(\frac{1}{2} + i\epsilon) & i(-i\chi)^{i\epsilon}\Gamma(\frac{1}{2} - i\epsilon) \\ \sqrt{\frac{H_{11}}{H_{22}}}(-i\chi)^{-i\epsilon}\Gamma(\frac{1}{2} + i\epsilon) & \sqrt{\frac{H_{11}}{H_{22}}}(-i\chi)^{i\epsilon}\Gamma(\frac{1}{2} - i\epsilon) \end{pmatrix}, \quad (47)$$

$$\mathfrak{T}_2(\chi) = \begin{pmatrix} -i(-i\chi)^{-i\epsilon}\Gamma(\frac{3}{2} + i\epsilon) & i(-i\chi)^{i\epsilon}\Gamma(\frac{3}{2} - i\epsilon) \\ \sqrt{\frac{H_{11}}{H_{22}}}(-i\chi)^{-i\epsilon}\Gamma(\frac{3}{2} + i\epsilon) & \sqrt{\frac{H_{11}}{H_{22}}}(-i\chi)^{i\epsilon}\Gamma(\frac{3}{2} - i\epsilon) \end{pmatrix}. \quad (48)$$

It is noted here that these expressions differ to those seen in Morini (2012) and Piccolroaz (2007) to incorporate the different branch cut used in this paper. It is now possible to find the asymptotic expansion of the members of Betti's identity from equation (44), using expressions (38) and (39), as $\chi \rightarrow \infty$

$$[\hat{\mathbf{U}}]^{+T} \mathcal{R} \hat{\mathbf{t}}^+ = \chi^{-1} \mathcal{Z}_1 \mathbf{K} + \chi^{-2} \mathcal{Z}_2 \mathbf{Y}_2 + \chi^{-3} \mathcal{Z}_3 \mathbf{Y}_3 + \mathcal{O}(\chi^{-4}), \quad \text{where } \text{Im}(\chi) \in (0, \infty), \quad (49)$$

$$\hat{\mathbf{Y}}^{-T} \mathcal{R} [\hat{\mathbf{u}}]^- = \chi^{-1} \mathcal{Z}_1 \mathbf{K} + \chi^{-2} \mathcal{Z}_2 \mathbf{Y}_2 + \chi^{-3} \mathcal{Z}_3 \mathbf{Y}_3 + \mathcal{O}(\chi^{-4}), \quad \text{where } \text{Im}(\chi) \in (-\infty, 0). \quad (50)$$

The matrices \mathcal{Z}_1 and \mathcal{Z}_2 are given by

$$\mathcal{Z}_1 = -\frac{H_{11}}{4s^+s^-(1+4\epsilon^2)} \begin{pmatrix} -\frac{(\beta-1)(1-2i\epsilon)}{E^2} & E^2(\beta+1)(1+2i\epsilon) \\ \frac{i(\beta-1)(1-2i\epsilon)}{E^2} & iE^2(\beta+1)(1+2i\epsilon) \end{pmatrix},$$

$$\mathcal{Z}_2 = -\frac{H_{11}}{4(1+4\epsilon^2)} \begin{pmatrix} -\frac{(\beta-1)(1-2i\epsilon)}{g^+s^-E^2} & \frac{E^2(\beta+1)(1+2i\epsilon)}{s^+g^-} \\ \frac{i(\beta-1)(1-2i\epsilon)}{g^+s^-E^2} & \frac{iE^2(\beta+1)(1+2i\epsilon)}{s^+g^-} \end{pmatrix},$$

where

$$E = e^{\epsilon\frac{\pi}{2}}, \quad s^\pm = \frac{(1+i)\sqrt{\pi}}{2\Gamma(\frac{1}{2} \pm i\epsilon)}, \quad g^\pm = \frac{(1-i)\sqrt{\pi}}{2\Gamma(\frac{3}{2} \pm i\epsilon)}.$$

Following the method of Morini (2012) (44) is written as a Riemann-Hilbert problem

$$\psi^+(\chi) - \psi^-(\chi) = -[\hat{\mathbf{U}}]^{+T} \mathcal{R} \langle \hat{\mathbf{p}} \rangle - \langle \hat{\mathbf{U}} \rangle^T \mathcal{R} [\hat{\mathbf{p}}], \quad (51)$$

using the Plemelj formula it is possible to find $\psi(\chi)$ using the formula

$$\psi^\pm(\chi) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi(\tau)}{\tau - \chi} d\tau. \quad (52)$$

The solution of this Riemann-Hilbert problem is given by

$$[\hat{\mathbf{U}}]^{+T} \mathcal{R} \hat{\mathbf{t}}^+ = \psi^+, \quad \text{where } \text{Im}(\chi) \in (0, \infty),$$

$$\hat{\mathbf{Y}}^{-T} \mathcal{R} [\hat{\mathbf{u}}]^- = \psi^-, \quad \text{where } \text{Im}(\chi) \in (-\infty, 0).$$

The asymptotic expansion of the Plemelj formula as $\chi \rightarrow \infty^\pm$ is given by

$$\psi^\pm(\chi) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi(\tau)}{\tau - \chi} d\tau = \chi^{-1} \mathbf{V}_1^\pm + \chi^{-2} \mathbf{V}_2^\pm + \mathcal{O}(\chi^{-3}). \quad (53)$$

Comparing the terms of this asymptotic expansion with the terms of the expansions (49) and (50) it is clear that $\mathbf{V}_j^\pm = \mathcal{Z}_j \mathbf{Y}_j$, where $\mathbf{Y}_1 = \mathbf{K}$. Using (53) it is easily seen that the stress intensity factor, \mathbf{K} , is given by

$$\mathbf{K} = \lim_{\chi \rightarrow \infty^\pm} \frac{1}{2\pi i} \mathcal{Z}_1^{-1} \int_{-\infty}^{\infty} \frac{\chi \left(-[\hat{\mathbf{U}}]^{+T}(\tau) \mathcal{R} \langle \hat{\mathbf{p}} \rangle(\tau) - \langle \hat{\mathbf{U}} \rangle^T(\tau) \mathcal{R} [\hat{\mathbf{p}}](\tau) \right)}{\tau - \chi} d\tau, \quad (54)$$

where the explicit expression for \mathcal{Z}_1^{-1} is given by

$$\mathcal{Z}_1^{-1} = \frac{2s^+ s^- (1 + 4\epsilon^2)}{H_{11}} \begin{pmatrix} \frac{E^2}{(\beta-1)(1-2i\epsilon)} & \frac{iE^2}{(\beta-1)(1-2i\epsilon)} \\ -\frac{1}{(\beta+1)(1+2i\epsilon)E^2} & \frac{i}{(\beta+1)(1+2i\epsilon)E^2} \end{pmatrix}.$$

Assuming that the loading disappears in the region of the crack tip the limit in (54) exists and therefore the expression for the stress intensity factor, \mathbf{K} , for the steady state agrees with that for the static case found in Morini (2012) (see equation (29)).

Now that an expression for the stress intensity factor has been found it is possible to determine the energy release rate. As the velocities considered in this paper are all sub-Rayleigh they are all subsonic and therefore, as stated in Yu (2000), equation (17) can be used with the steady state-coordinates with Δ still an arbitrary length scale. Using (18) the following expression is obtained for the ERR in orthotropic materials

$$G = \frac{H_{11}(1 - \beta^2)|K|^2}{4}. \quad (55)$$

3.2 General Expression for the Coefficients of the Higher Order Terms

Using the asymptotic expansions (49), (50) and the corresponding terms of (53) a general expression for the j th coefficient of the asymptotic expansions, \mathbf{Y}_j , is found

$$\mathbf{V}_j^\pm = \lim_{\chi \rightarrow \infty^\pm} \left[\frac{\chi^j (-1)^{j-1}}{2\pi i (j-1)!} \int_{-\infty}^{\infty} \psi(\tau) \frac{d^{j-1}}{d\chi^{j-1}} \left(\frac{\chi^{j-1}}{\tau - \chi} \right) d\tau \right]. \quad (56)$$

This gives a general expression for the coefficients of the asymptotic expansion of the displacement and stress fields as

$$\mathbf{Y}_j = \lim_{\chi \rightarrow \infty^\pm} \frac{1}{2\pi i} \mathcal{Z}_j^{-1} \int_{-\infty}^{\infty} \tau^{j-1} ([\hat{\mathbf{U}}]^{+T}(\tau) \mathcal{R} \langle \hat{\mathbf{p}} \rangle(\tau) + \langle \hat{\mathbf{U}} \rangle^T(\tau) \mathcal{R} [\hat{\mathbf{p}}](\tau)) \left(\frac{\chi}{\chi - \tau} \right)^j d\tau. \quad (57)$$

If the loading is applied in such a way that the limit exists it is easily seen that equation (57) is equivalent to (30).

However, in general, the limit in (57) cannot be computed directly for $j \geq 2$. In this case an alternate method must be used in order to compute \mathbf{Y}_j .

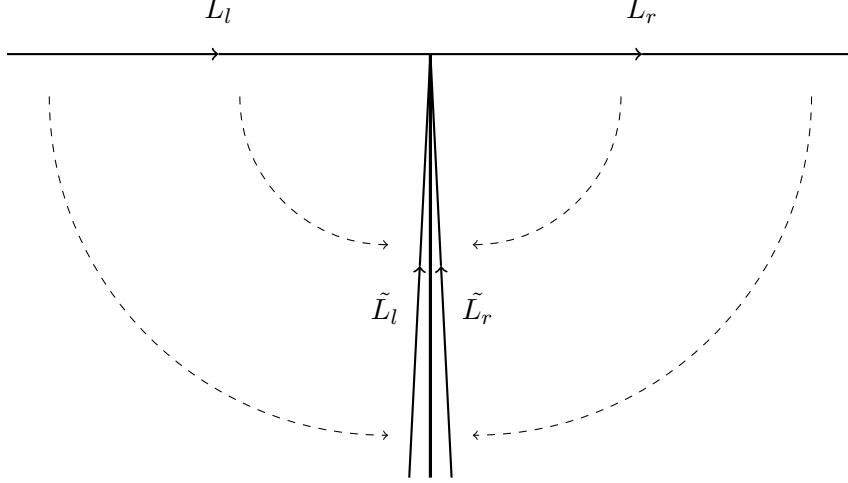


Figure 2: Integration Shift in the χ -Plane

As the function \mathbf{p} only exists on the negative real x -axis its Fourier transform is analytic in the lower half χ -plane. Therefore, $[\hat{\mathbf{p}}]$ and $\langle \hat{\mathbf{p}} \rangle$ are also analytic in the lower-half plane. As long as the applied loading \mathbf{p} vanishes within a region of the crack tip it is clearly seen that $[\hat{\mathbf{p}}]$ and $\langle \hat{\mathbf{p}} \rangle$ decay exponentially as they tend to $-i\infty$. It is also known that both $[\hat{\mathbf{U}}]^+$ and $\langle \hat{\mathbf{U}} \rangle$ are analytic in the lower-half plane apart from the negative imaginary axis.

Using this information the contour of integration is moved to be either side of the negative imaginary axis, as shown in Figure 2. Equation (57) now becomes

$$\mathbf{Y}_j = \lim_{\chi \rightarrow \infty \pm} \left(-\frac{1}{2\pi i} \mathcal{Z}_j^{-1} \left[\int_{\tilde{L}_l} \tau^{j-1} \psi(\tau) \left(\frac{\chi}{\chi - \tau} \right)^j d\tau - \int_{\tilde{L}_r} \tau^{j-1} \psi(\tau) \left(\frac{\chi}{\chi - \tau} \right)^j d\tau \right] \right). \quad (58)$$

The limit of (58) can be taken to give

$$\mathbf{Y}_j = \frac{1}{2\pi i} \mathcal{Z}_j^{-1} \int_{-i\infty}^0 \tau^{j-1} [-\psi(\tau)] d\tau, \quad (59)$$

where $[\psi(\tau)]$ refers to the jump of the function ψ over the negative imaginary axis.

The expression (59) can be simplified further by considering the continuity of (38) and (39). In order to do this (38) and (39) are rewritten in the following way

$$[\hat{U}]^+(\chi) = \frac{1}{|\chi|} \begin{pmatrix} 0 & -i\beta \text{sign}(\chi) \sqrt{H_{11}H_{22}} \\ i\beta \text{sign}(\chi) \sqrt{H_{11}H_{22}} & 0 \end{pmatrix} \hat{\mathbf{r}}^-(\chi) - \frac{1}{|\chi|} \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix} \hat{\mathbf{r}}^-(\chi), \quad (60)$$

$$\langle \hat{U} \rangle(\chi) = \frac{1}{2|\chi|} \begin{pmatrix} 0 & i\gamma \text{sign}(\chi) \sqrt{H_{11}H_{22}} \\ -i\gamma \text{sign}(\chi) \sqrt{H_{11}H_{22}} & 0 \end{pmatrix} \hat{\mathbf{r}}^-(\chi) - \frac{1}{2|\chi|} \begin{pmatrix} \delta_1 H_{11} & 0 \\ 0 & \delta_2 H_{22} \end{pmatrix} \hat{\mathbf{r}}^-(\chi). \quad (61)$$

The first term of both of these expressions simplify to give, respectively

$$\frac{1}{\chi} \begin{pmatrix} 0 & -i\beta \sqrt{H_{11}H_{22}} \\ i\beta \sqrt{H_{11}H_{22}} & 0 \end{pmatrix} \hat{\mathbf{r}}^-(\chi), \quad \text{and} \quad \frac{1}{2\chi} \begin{pmatrix} 0 & -i\gamma \sqrt{H_{11}H_{22}} \\ i\gamma \sqrt{H_{11}H_{22}} & 0 \end{pmatrix} \hat{\mathbf{r}}^-(\chi).$$

Both of these terms are analytic in the lower half-plane and therefore continuous across the negative imaginary axis. For this reason they do not contribute to the general expression for the asymptotic coefficients, (59).

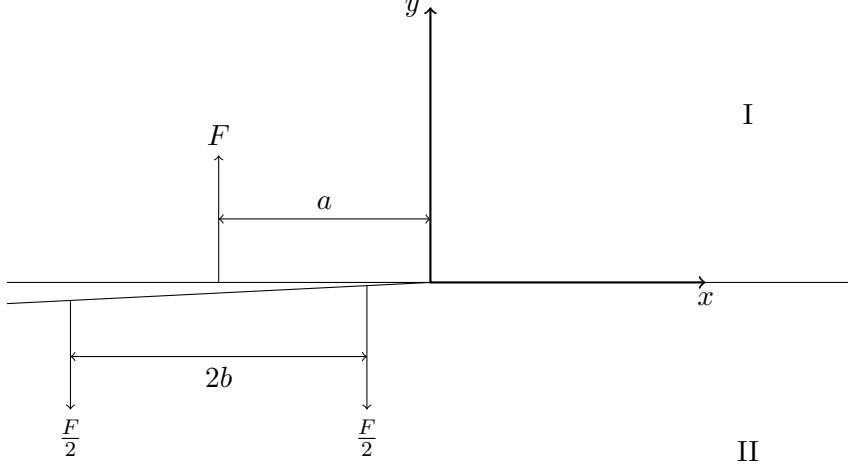


Figure 3: Loading

A simpler expression for the asymptotic coefficients is now found

$$\mathbf{Y}_j = \frac{1}{2\pi i} \mathcal{Z}_j^{-1} \int_{-\infty}^0 \tau^{j-1} [-\phi(\tau)] d\tau, \quad (62)$$

where ϕ is given by the expression

$$\phi(\tau) = \text{Re}(\mathbf{H}) \hat{\mathbf{Y}}^-(\tau) \mathcal{R}[\hat{\mathbf{p}}](\tau) + \text{Re}(\mathbf{W}) \hat{\mathbf{Y}}^-(\tau) \mathcal{R}[\hat{\mathbf{p}}](\tau).$$

4 A Specific Example

A specific example for computing the stress intensity factors for orthotropic materials is now considered. The loading on the crack faces is given by a point force of magnitude F acting on the upper crack face a distance a behind the crack tip and two point forces, both of magnitude $F/2$, acting on the lower crack face a distance b away from the point force acting upon the upper crack face. The loading moves at the same speed and in the same direction that the crack is propagating. This is shown in Figure 3. The forces are represented mathematically using the Dirac delta function (Piccolroaz, 2009)

$$p_+(x) = -F\delta(x+a), \quad p_-(x) = -\frac{F}{2}\delta(x+a+b) - \frac{F}{2}\delta(x+a-b). \quad (63)$$

It is now possible to decompose the loading into its symmetric and skew-symmetric components

$$\begin{aligned} \langle p \rangle(x) &= \frac{1}{2}[p_+(x) + p_-(x)] = -\frac{F}{2}\delta(x+a) - \frac{F}{4}\delta(x+a+b) - \frac{F}{4}\delta(x+a-b), \\ [p](x) &= p_+(x) - p_-(x) = -F\delta(x+a) + \frac{F}{2}\delta(x+a+b) + \frac{F}{2}\delta(x+a-b). \end{aligned} \quad (64)$$

In order to compute the stress intensity factors the Fourier transforms of the skew-symmetric and symmetric parts of the loading are required. These are given by

$$\langle \hat{p} \rangle(\chi) = -\frac{F}{2}e^{-i\chi a} - \frac{F}{4}e^{-i\chi(a+b)} - \frac{F}{4}e^{-i\chi(a-b)}, \quad (65)$$

$$[\hat{p}](\chi) = -Fe^{-i\chi a} + \frac{F}{2}e^{-i\chi(a+b)} + \frac{F}{2}e^{-i\chi(a-b)}. \quad (66)$$

It is now possible to compute expressions for the first and second order asymptotic coefficients, \mathbf{K} and \mathbf{Y}_2 , using expressions (54) and (57) respectively.

To find an expression for \mathbf{K} equation (54) is used. The solution is split into the parts corresponding to the symmetric and anti-symmetric parts of the loading, denoted K^S and K^A respectively

$$K^S = F \frac{E^2}{(1-\beta)} \sqrt{\frac{H_{22}}{H_{11}}} \sqrt{\frac{2}{\pi}} a^{-\frac{1}{2}-i\epsilon} \left[\frac{1}{2} + \frac{1}{4}(1+b/a)^{-\frac{1}{2}-i\epsilon} + \frac{1}{4}(1-b/a)^{-\frac{1}{2}-i\epsilon} \right], \quad (67)$$

$$K^A = F \frac{E^2 \delta_2}{(1-\beta)} \sqrt{\frac{H_{22}}{H_{11}}} \sqrt{\frac{2}{\pi}} a^{-\frac{1}{2}-i\epsilon} \left[\frac{1}{2} - \frac{1}{4}(1+b/a)^{-\frac{1}{2}-i\epsilon} - \frac{1}{4}(1-b/a)^{-\frac{1}{2}-i\epsilon} \right]. \quad (68)$$

It is noted that for the loading considered in figure (3) that the limit in (57) does not exist and therefore equation (62) must be used with $j = 2$. Once again the coefficient is split into symmetric and anti-symmetric parts, given by

$$Y_2^S = F \frac{E^2}{(\beta-1)} \sqrt{\frac{H_{22}}{H_{11}}} \sqrt{\frac{2}{\pi}} a^{-\frac{3}{2}-i\epsilon} \left[\frac{1}{2} + \frac{1}{4}(1+b/a)^{-\frac{3}{2}-i\epsilon} + \frac{1}{4}(1-b/a)^{-\frac{3}{2}-i\epsilon} \right], \quad (69)$$

$$Y_2^A = F \frac{E^2 \delta_2}{(\beta-1)} \sqrt{\frac{H_{22}}{H_{11}}} \sqrt{\frac{2}{\pi}} a^{-\frac{3}{2}-i\epsilon} \left[\frac{1}{2} - \frac{1}{4}(1+b/a)^{-\frac{3}{2}-i\epsilon} - \frac{1}{4}(1-b/a)^{-\frac{3}{2}-i\epsilon} \right]. \quad (70)$$

Having computed expressions for the stress intensity factors it is now possible to calculate the energy release rate for two given materials. Material I is the piezoceramic Barium Titanate which has material properties $C_{11} = 120.3\text{GPa}$, $C_{22} = 120.3\text{GPa}$, $C_{12} = 75.2\text{GPa}$, $C_{66} = 21\text{GPa}$ and $\rho = 6,020\text{kgm}^{-3}$. Material II is set as Aluminium which has elastic constants $C_{11} = 107.3\text{GPa}$, $C_{22} = 107.3\text{GPa}$, $C_{12} = 60.9\text{GPa}$, $C_{66} = 28.3\text{GPa}$ and $\rho = 2,700\text{kgm}^{-3}$. For the purpose of calculations, a is set as 1 in this paper. The velocity is normalised by dividing by c_R , the lowest of the two Rayleigh wave speeds for the given materials, which for the materials considered is that of Barium Titanate. The energy release rate is non-dimensionalised in the results shown. G is normalised in the following manner: $GC_{66}^{(1)}/F^2$. Here, $C_{66}^{(1)}$ is taken as the value of C_{66} for the material above the crack.

Fig. 4 shows the normalised ERR and Fig. 5 indicates the normalised energy release rates, G^S and G^A , corresponding to K^S and K^A , respectively. Both components are normalised by the total energy release rate corresponding to $K = K^S + K^A$, given by G .

The graph in Fig. 4 clearly shows that ERR increases as the velocity increases for all values of b/a and approaches infinity as the velocity approaches the Rayleigh wave speed, as expected. Moreover, the energy release rate is higher for larger values of b/a , that is, as the asymmetry of the loading increases. Therefore, symmetric loading is energetically more beneficial than any choice of asymmetric one.

Fig. 5 show that the contribution of the asymmetric part of the loading to the overall energy release rate is small compared to the contribution of the symmetric part for the lower crack velocities. However, when the crack is moving at velocities close to the Rayleigh wave speed the asymmetric nature of the loading begins to play a more crucial role in the calculations. This fact is also highlighted in the left hand graph of Fig. 6. Interestingly, the energy release rate corresponding to the symmetric part of the loading is higher than the overall ERR for the physical loading, apart from when the crack velocity is very close to the Rayleigh wave speed. The behaviour of the symmetric part of G near the Rayleigh wave speed is particularly noteworthy as it can be seen that for asymmetric loading (when $b/a > 0$) the ERR stops increasing and starts decreasing. Note that, for the symmetrical load ($b/a = 0$), the energy release does not depend on the crack speed. This was previously observed for isotropic and anisotropic bimetals in (Wu, 1990; Yang, 1991).

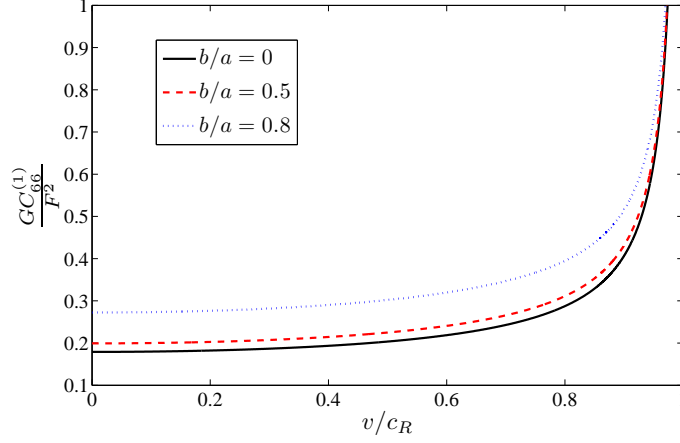


Figure 4: The normalised ERR, as a function of the velocity, for different positions of the self-balanced point forces applied to the crack surfaces, described by the ratio b/a in Fig. 1.

Fig. 6a shows the ratio of the mode 2 contribution of the stress intensity factor, K , to the mode 1 component. For the symmetric case the mode 2 component is 0 for all values of the velocity, as one would expect, whereas for asymmetric symmetry the ratio is initially negative and then at a certain velocity the sign of the ratio changes. This is connected to the fact that for a determinate value of the propagation speed there is a change in the sign of the Dundurs parameter, β (which only depends on the elastic constants of the materials). It is therefore possible to obtain a characteristic value for v , depending only on the material properties. This may lead to a possible change in the fracture mechanism or indicate a possible redirection of the interface crack propagation, for example kinking or branching. Note that there are other investigations, both theoretical and experimental, demonstrating an existence of a specific sub-Rayleigh velocity which is related to the stability of the crack propagation (Obrezanova, 2002b,a).

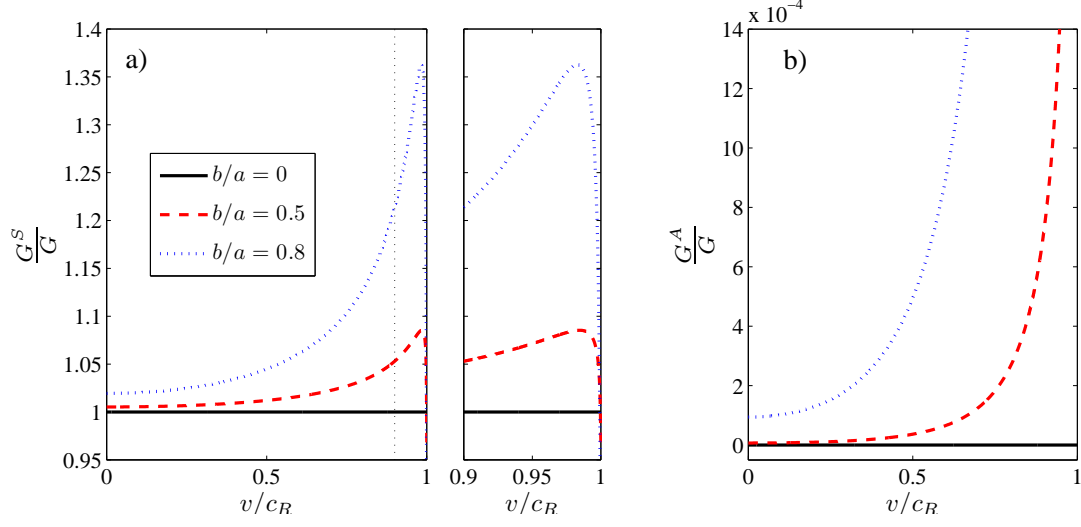


Figure 5: a) The ratio G^S/G , as a function of velocity, for different values of b/a , with particular attention being given to the behaviour near c_R . b) The ratio G^A/G with a different scale on the vertical axis.

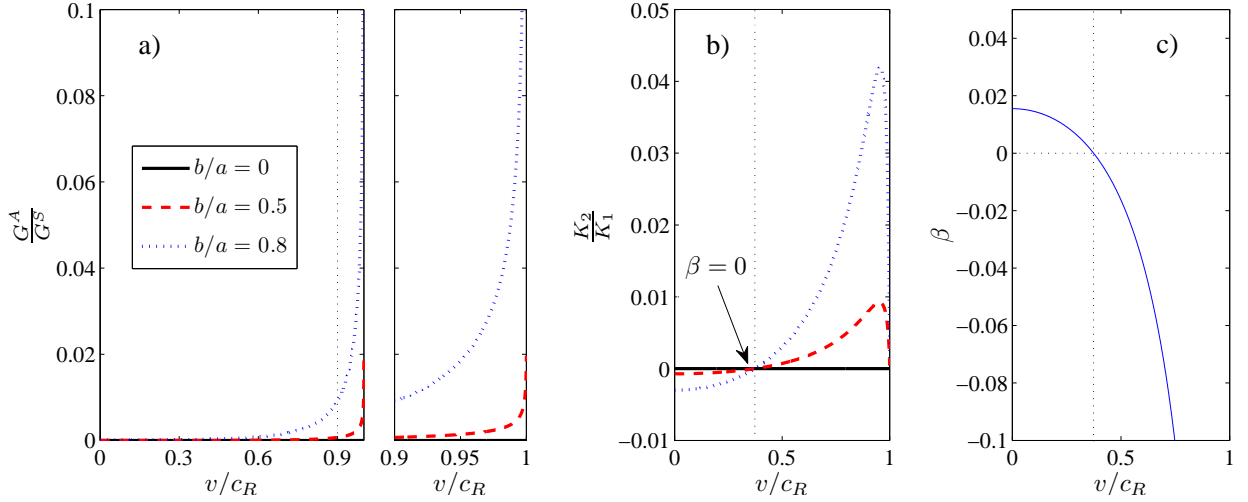


Figure 6: a) The ratio G^A/G^S , as a function of the velocity, for different values of b/a with particular attention being given to the behaviour near c_R . b) The ratio K_2/K_1 , as a function of the velocity, for different values of b/a . c) The value of β as a function of the velocity.

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A Orthotropic Stroh Matrices

For orthotropic materials the matrices \mathbf{Q} , \mathbf{R} and \mathbf{T} are given by

$$\mathbf{Q} = \begin{pmatrix} C_{11} - \rho v^2 & 0 \\ 0 & C_{66} - \rho v^2 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 0 & C_{12} \\ C_{66} & 0 \end{pmatrix}, \mathbf{T} = \begin{pmatrix} C_{66} & 0 \\ 0 & C_{22} \end{pmatrix} \quad (71)$$

Previously, expressions were found for the Stroh matrices for an orthotropic bimaterial with a crack propagating at speed v (Yang, 1991). The following parameters are defined in the same manner as Yang (1991)

$$\kappa_{\gamma\beta} = \frac{C_{\gamma\beta}}{C_{66}} \quad \alpha_1 = \sqrt{1 - \frac{\rho v^2}{C_{11}}} \quad \alpha_2 = \sqrt{1 - \frac{\rho v^2}{C_{66}}}$$

and the variables

$$\xi = \alpha_1 \alpha_2 \sqrt{\frac{\kappa_{11}}{\kappa_{22}}} \text{ and } s = \frac{\alpha_2^2 + \kappa_{11} \kappa_{22} \alpha_1^2 - (1 + \kappa_{12})^2}{2\alpha_1 \alpha_2 \sqrt{\kappa_{11} \kappa_{22}}}$$

It is seen that the eigenvalues, with positive imaginary part, of the equation

$$(\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T})\mathbf{A} = 0 \quad (72)$$

for the matrices from (71) are given by

$$p_{1,2} = \begin{cases} i\sqrt{\xi} \left(\sqrt{\frac{s+1}{2}} \pm \sqrt{\frac{s-1}{2}} \right), & \text{for } s \geq 1 \\ \sqrt{\xi} \left(\pm \sqrt{\frac{1-s}{2}} + i\sqrt{\frac{1+s}{2}} \right), & \text{for } -1 < s < 1 \end{cases} \quad (73)$$

Using the same normalisation as used in Yang (1991) the matrices \mathbf{A} and \mathbf{L} are given by

$$\mathbf{A} = \begin{pmatrix} 1 & -\lambda_2^{-1} \\ -\lambda_1 & 1 \end{pmatrix} \quad (74)$$

$$\mathbf{L} = C_{66} \begin{pmatrix} p_1 - \lambda_1 & 1 - p_2\lambda_2^{-1} \\ \kappa_{12} - \kappa_{22}p_1\lambda_1 & \kappa_{22}p_2 - \kappa_{12}\lambda_2^{-1} \end{pmatrix} \quad (75)$$

where

$$\lambda_\mu = \frac{\kappa_{11}\alpha_1^2 + p_\mu^2}{(1 + \kappa_{12})p_\mu}$$

It is now possible to find an expression for the hermitian matrix \mathbf{B} given by

$$\mathbf{B} = i\mathbf{A}\mathbf{L}^{-1} = \frac{1}{C_{66}F} \begin{pmatrix} \kappa_{22}\alpha_2^2\sqrt{2(1+s)/\xi} & i(\kappa_{22} - \kappa_{12}\alpha_2^2/\xi) \\ -i(\kappa_{22} - \kappa_{12}\alpha_2^2/\xi) & \kappa_{22}\sqrt{2\xi(1+s)} \end{pmatrix} \quad (76)$$

where F is the generalized Rayleigh wave function given by

$$F = \kappa_{22}(\kappa_{22}\xi - 1 + \alpha_2^2) - \kappa_{12}^2\alpha_2^2/\xi$$

It is possible to find the Rayleigh wave speed of a material by setting $F = 0$.

The bimaterial matrix \mathbf{H} can now be computed

$$\mathbf{H} = \begin{pmatrix} H_{11} & -i\beta\sqrt{H_{11}H_{22}} \\ i\beta\sqrt{H_{11}H_{22}} & H_{22} \end{pmatrix} \quad (77)$$

From (76) it is seen that

$$\begin{aligned} H_{11} &= \left[\frac{\kappa_{22}\alpha_2^2\sqrt{2(1+s)/\xi}}{C_{66}R} \right]_I + \left[\frac{\kappa_{22}\alpha_2^2\sqrt{2(1+s)/\xi}}{C_{66}R} \right]_{II} \\ H_{22} &= \left[\frac{\kappa_{22}\sqrt{2\xi(1+s)}}{C_{66}R} \right]_I + \left[\frac{\kappa_{22}\sqrt{2\xi(1+s)}}{C_{66}R} \right]_{II} \\ \beta\sqrt{H_{11}H_{22}} &= \left[\frac{\kappa_{22} - \kappa_{12}\alpha_2^2/\xi}{C_{66}R} \right]_{II} - \left[\frac{\kappa_{22} - \kappa_{12}\alpha_2^2/\xi}{C_{66}R} \right]_I \end{aligned}$$

In order to compute the weight functions the eigenvalues and eigenvectors of (9) are required. Using the representation (77) it is found that

$$\mathbf{w} = \begin{pmatrix} -\frac{i}{2} \\ \frac{1}{2}\sqrt{\frac{H_{11}}{H_{22}}} \end{pmatrix} \quad \epsilon = \frac{1}{2\pi} \ln \left(\frac{1-\beta}{1+\beta} \right) \quad (78)$$

Another key component for calculating the weight functions is the bimaterial matrix $\mathbf{W} = \mathbf{B}_I - \mathbf{B}_{II}^-$. Using (76) it is seen that

$$\mathbf{W} = \sqrt{H_{11}H_{22}} \begin{pmatrix} \delta_1 \sqrt{\frac{H_{11}}{H_{22}}} & i\gamma \\ -i\gamma & \delta_2 \sqrt{\frac{H_{22}}{H_{11}}} \end{pmatrix} \quad (79)$$

where

$$\begin{aligned} \gamma &= \frac{\left[\frac{\kappa_{22} - \kappa_{12}\alpha_2^2/\xi}{C_{66}R} \right]_I + \left[\frac{\kappa_{22} - \kappa_{12}\alpha_2^2/\xi}{C_{66}R} \right]_{II}}{\sqrt{H_{11}H_{22}}} \\ \delta_1 &= \frac{\left[\frac{\kappa_{22}\alpha_2^2\sqrt{2(1+s)/\xi}}{C_{66}R} \right]_I - \left[\frac{\kappa_{22}\alpha_2^2\sqrt{2(1+s)/\xi}}{C_{66}R} \right]_{II}}{H_{11}} \\ \delta_2 &= \frac{\left[\frac{\kappa_{22}\sqrt{2\xi(1+s)}}{C_{66}R} \right]_I - \left[\frac{\kappa_{22}\sqrt{2\xi(1+s)}}{C_{66}R} \right]_{II}}{H_{22}} \end{aligned}$$